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New Analytical Method for Solving Nonlinear Fraction Partial Differentional Equations

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Abstract— Similar to Kantorovich method for variations of calculus, a new method called the fractional series expansion is proposed to solve nonlinear fractional differential equations. The solution is assumed to be an infinite series of separated functions of independent variables. The solution procedure is elucidated by two examples. The fractional generalized coupled MKDV and KDV equation is used as another example to show that its solution depends strongly upon its initial conditions, a special condition is given when no solution exists for the discussed problem.

Index Terms— Riemann Liouville derivative, Caputo derivative, fractional generalized coupled MKDV and KDV equation, series solution.

1 Introduction

THE seeds of fractional calculus were planted over 300 years ago, and now a forest of its applications in various fields is formed, this is because that differential equations involving derivatives of non-integer order can be adequate models for various physical phenomena [1, 2] in especially discontinuous media. Nobel Laureate Gerardus 't Hooft [3] once remarked that discrete space–time is the most radical and logical viewpoint of reality, and fractal theory and fractional calculus are best candidates for description of phenomena in discrete space–time, it is interesting to find that the fractional order is equivalent to its fractional dimensions[4].

Recently some analytical methods were appeared in open literature for fractional calculus, among which the variational iteration method (VIM) [5], and the homotopy perturbation method [6], the exp-function method[7-9], the fractional complex transform[10], and local fractional integral transforms including the Yang-Fourier transform and Yang-Laplace transform[11] have been caught much attention. In our paper we will suggest a novel method called the fractional series expansion to solve fractional differential equations, using the Caputo time fractional derivative operator, our new method is simple but effective.

2 Preliminaries and Notations

Fractional differential equations have excited, in recent years, a considerable interest both in mathematics and in applications. They were used in modeling of many physical and chemical processes and engineering (see, e.g., [12-16]).

Here, we mention the basic definitions of the Caputo fractional-order integration and differentiation, which are used in the upcoming paper and play the most important role in the theory of differential and integral equation of fractional order. The main advantages of Caputo approach are the initial conditions for fractional differential equations with the Caputo derivatives taking on the same form as for integer order differential equations. In this section, we give some basic. For more details for definitions and properties of the fractional calculus theory which will be used further in this work see [1].

We shall introduce a modified fractional differential operator D^{α} proposed by M. Caputo in his work on the theory of viscoelasticity [1].

Definition. For m to be the smallest integer that exceeds α, the Caputo time fractional derivative operator of order $\alpha > 0$ is defined as

$$f^{\alpha}(x)=\frac{1}{\Gamma(1-\alpha)}\int_0^x (x-\xi)^{-\alpha}\frac{d}{dx}f(\xi)d\xi, 0<\alpha<1$$

3 New Analytical Method

We consider a fractional partial differential equation of the form

$$D_t^{\alpha}u = Lu, \tag{1}$$

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with initial condition

$$u(x,0)=f(x), \tag{2}$$

where L is a differential operator containing only derivatives with respect to x. Hinted by the Kantorovich method in calculus of variations, the solution can be assumed as multi-term separated functions of independent variables t and x:

$$u(x,t)=\sum_n h_n(t)g_n(x) \tag{3}$$

for the case when α is an integer. Hereby $h_n(t)$ and $g_n(x)$ are unknown functions to be further determined later.

We assume that the solution can be expressed in the form

$$u(x,t)=\sum_{n=0}^{\infty} t^{n\alpha} g_n(x). \tag{4}$$

Substituting Eq. (4) into Eq. (1), by simple calculation, we have

$$\sum_{n=0}^{\infty}\frac{\Gamma(1+(n+1)\alpha)}{\Gamma(1+n\alpha)}t^{n\alpha}\, g_{n+1}(x)=\sum_{n=0}^{\infty}t^{n\alpha}(Lg_n)(x) \tag{5}$$

Comparing coefficients we obtain

$$g_0(x)=f(x) \tag{6}$$

and

$$g_{n+1}(x)=\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)}(Lg_n)(x). \tag{7}$$

Therefore, we have

$$g_n=\frac{1}{\Gamma(1+n\alpha)}L^n f\ . \tag{8}$$

and the following series solution

$$u(x,t)=\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}(L^n f)(x). \tag{9}$$

Formula (9) is true only when the operator L is linear. If L is nonlinear one has to compare coefficients in the usual way.

4 EXAMPLES

Example 1. Consider the one dimension fraction heat conduction equation

$$D_t^{\alpha}u+Au_{xx}=0,$$

with the initial condition

$$u(x,0)=\sin x\ .$$

Where A is a constant and $t\geq 0, 0<\alpha<1$.

By a simple calculation, we obtain

$$(Lf)(x)=A\sin x,\ (L^2 f)(x)=A^2\sin x\ ,\dots .$$

Then

$$u(x,t)=f(x)+\frac{t^{\alpha}}{\Gamma(1+\alpha)}(Lf)(x)+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}(L^2 f)(x)+\dots$$
$$=\sin x+A\frac{t^{\alpha}}{\Gamma(1+\alpha)}\sin x+A^2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\sin x+\dots$$

is some kind of approximate solution .

Example 2. Consider the following fractional heat equation

$$D_t^{\alpha}u+u_{xx}+u^2=0, \tag{10}$$

with the initial condition

$$u(x,0)=x^2. \tag{11}$$

Assume the solution can be expressed in the form

$$u(x,t)=\sum_{n=0}^{\infty}t^{n\alpha}u_n(x). \tag{12}$$

By a simple calculation, we obtain

$$u^2=\sum_{n=0}^{\infty}t^{n\alpha}\sum_{i+j=n}u_i u_j, \tag{13}$$

and

$$u_{xx}=\sum_{n=0}^{\infty}t^{n\alpha}u_n''. \tag{14}$$

Substituting Eqs. (12)-(14) into Eq. (10), and comparing coefficients, we obtain

$$u_0(x)=x^2 \tag{15}$$

and

$$\frac{\Gamma(1+(n+1)\alpha)}{\Gamma(1+n\alpha)}u_{n+1}=-\sum_{i+j=n}u_i u_j-u_n''. \tag{16}$$

Using this formula, we find sequentially that

$$u_1(x)=\frac{\Gamma(1)}{\Gamma(1+\alpha)}(u_0^2+u_0'')=-\frac{\Gamma(1)}{\Gamma(1+\alpha)}(x^4+2),$$

$$u_2(x) = -\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(2u_0 \quad u_1 + u_1''\right) = \frac{1}{\Gamma(1+2\alpha)}\left(2x^6 + 16x^2\right),$$

$$u_3(x) = \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(-2u_0 \quad u_2 - u_1^2 - u_2''\right)$$

$$= \frac{1}{\Gamma(1+3\alpha)} \cdot \left\{\left(-32x - 12x^5 - 32x^4 - 4x^8\right) - \frac{\Gamma(1+2\alpha)}{\left(\Gamma(1+\alpha)\right)^2}\left(x^8 + 4x^4 + 4\right)\right\} \tag{17}$$

We, therefore, obtain the following series solution

$$u(x,t) = x^2 - \frac{t^{\alpha}}{\Gamma(1+\alpha)}\left(x^4 + 2\right) + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\left(2x^6 + 16x^2\right) - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\left\{\left(-32x - 12x^5 - 32x^4 - 4x^8\right) - \frac{\Gamma(1+2\alpha)}{\left(\Gamma(1+\alpha)\right)^2}\left(x^8 + 4x^4 + 4\right)\right\} + \ldots \tag{18}$$

The example can be also solved using the homotopy perturbation method [6]. The homotopy equation can be constructed as

$$D_t^{\alpha} u + p(u_{xx} + u^2) = 0 \tag{19}$$

where p is a homotopy parameter. Assume that the solution of Eq. (19) can be expressed as

$$u = u_0 + pu_1 + p^2 u_2 + \cdots . \tag{20}$$

Following the homotopy perturbation method, we have

$$u_1 = \left(-2 - x^4\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)}, \tag{21}$$

Other components can be solved sequently, hereby we write down only the first-order approximate solution, which is

$$u(x,t) = u_0 + u_1 = x^2 + \left(-2 - x^4\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \ldots \tag{22}$$

It is obvious that the first few terms agree with those in Eq. (18).

In case $\alpha = 1$, the example can be written as

$$u_t + u_{xx} + u^2 = 0, \qquad u(x,0) = x^2 \tag{23}$$

This equation can be solved by the variational iteration method [5]:

$$u_{n+1} = u_n - \int_0^t (u_{nt} + u_{nxx} + u_n^2)dt \tag{24}$$

If we begin with $u_0(x,t) = x^2$, we have the following first-order approximate solution

$$u_1 = x^2 - \int_0^t (2 + x^4)dt = x^2 - (2 + x^4)t \tag{25}$$

$$u_2 = \frac{t^2}{2}\left(16x^2 + 2x^6\right) - \frac{t^3}{6}\left(40 + 100x^4 + 2x^8\right) \tag{26}$$

So we get to

$$u(x,t) = x^2 - (2 + x^4)t + \frac{t^2}{2}\left(16x^2 + 2x^6\right) - \frac{t^3}{6}\left(40 + 100x^4 + 2x^8\right) + \ldots \tag{27}$$

So we get the same result as in (18) when $\alpha = 1$, and this is a good agreement.

The next example shows that the solution of the fractional generalized coupled MKDV and KDV equation depend on the choice of the initial condition.

Example 3. Consider the fractal derivative generalized coupled MKDV and KDV equation

$$D_t^{\alpha} t + \left(a + bu + cu^2\right)x + eu_{xxx} = 0,$$

where a, b, c, e are all constants, with the initial condition

$$u(x,0) = x^2.$$

We try to find series solutions of the form

$$u(x,t) = \sum_{n=0}^{\infty} t^{n\alpha} u_n(x).$$

We obtain

$$D_t^{\alpha} u = \sum_{n=0}^{\infty} \frac{\Gamma(1 + (n+1)\alpha)}{\Gamma(1 + n\alpha)} t^{n\alpha} u_{n+1}(x)$$

$$u\, u_x = \sum_{n=0}^{\infty} t^{n\alpha} \sum_{i+j=n} u_i u_j',$$

$$u^2 u_x = \sum_{n=0}^{\infty} t^{n\alpha} \sum_{i+j+k=n} u_i \quad u_j \quad u_k',$$

$$u_{xxx} = \sum_{n=0}^{\infty} t^{n\alpha} u_n'''.$$

Comparing coefficients we obtain $u_0(x) = x^2$ and then recursively

$$\frac{\Gamma(1+(n+1)\alpha)}{\Gamma(1+n\alpha)} u_{n+1} = -a u_n' - b \sum_{i+j=n} u_i u_j' - c \sum_{i+j+k=n} u_i u_j u_k' - e u_n'''.$$

The important question is : Does the series converge? In general, we expect that the series will not converge for any (x,t) except for $t = 0$. If $\alpha = 1$ this is well known from the Cauchy-Kovalevsky theory. The Cauchy-Kovalevsky theorem gives conditions which guarantee the convergence of the series solution. In our example (with $\alpha = 1$) these conditions are not satisfied. As an example consider the special case $u_t = u u_x + u_{xxx}$, with the initial condition $u(x,0) = x^2$.

Then the recursive formula for u_n is

$$(n+1) u_{n+1} = \sum_{i+j=n} u_i \quad u_j' + u_n'''.$$

Using this formula we find

$$u_1(x) = 2x^3$$
$$u_2(x) = 6 + 5x^4$$
$$u_3(x) = 44x + 14x^5$$
$$u_4(x) = 252x^2 + 42x^6$$
$$u_5(x) = 1304x^3 + 132x^7$$
$$u_6(x) = 1348 + 6380x^4 + 429x^8$$

One can show that $u_n(x)$ is a polynomial of degree $n+2$. The coefficients are all > 0 and the coefficient of x^{n+2} is at least 1. If $p(x) = \sum_{i=0}^{m} a_i x^i$, $q(x) = \sum_{i=0}^{m} b_i x^i$ are polynomials with nonnegative coefficients we write $p \succ q \; if \; a_i \geq b_i$ for all i. It follows that

$$(n+1) u_{n+1} \succ (n+2)(n+1) n x^{n-1}.$$

Then

$$(n+2)(n+1) u_{n+2} \succ (n+1) u_{n+1}''' \succ (n+2)(n+1) n (n-1)(n-2)(n-3) x^{n-4}.$$

If $n = 3k-2$ we obtain after *k* steps

$$(4k-2)(4k-3)\ldots(3k-1) u_{4k-2} \geq (3k)!$$

which implies

$$u_{4k-2}(x) \geq \frac{(3k!)(3k-2)!}{(4k-2)!} \quad \text{for } x \geq 0.$$

We obtain from this that the series diverges to ∞ for every $x \geq 0, t > 0$.

5. Conclusion

We suggest a fractional series expansion for fractional calculus, two examples are given revealing that the solution process is simple and accessible to non-mathematicians. Also we show that the solution of the fractional generalized coupled MKDV and KDV equation depend on the choice of the initial condition, i.e. we showed that the fraction partial differential equation $D_t^{\alpha} t + (a + bu + cu^2) x + e u_{xxx} = 0$ with the initial condition $u(x,0) = f(x)$ has no solution to show that we use The Cauchy-Kovalevsky theorem.

6. Discussion

The new method is valid for the case when the series is convergent, it is generally difficult to prove the obtained series is convergent. To overcome the problem, we also suggest an asymptotic fractional series expansion as follows

$$u(x,t) = t^{\alpha} u_1(x)$$

$$u(x,t) = (\sum_n c_n t^{n\alpha}) u_1(x)$$

$$u(x,t) = (\sum_n c_n t^{n\alpha})(\sum_m u_m(x))$$

where u_i and c_i (i=1,2,3...) are, respectively, unknown function and constant to be further determined approximately, we will discuss various cases in a forthcoming paper.

References

[1] Podlubny.I, Fractional Differential Equations: An Introduction to Fractional

Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications to Methods of Their Solution and Some of Their Applications, Academic Press, New York, NY, USA, 1999.

[2] Hooft. G, A confrontation with infinity, Nobel lecture, December 8, 1999.

[3] Gorenflo.R, Mainardi. F, Fractional calculus: integral and differential equations of fractional order," in Fractals Fractional Calculus in Continuum Mechanics, A. Carpinteri and F. Mainar, Eds., pp. 223–276, Springer, New York, NY, USA, 1997.

[4] He .J.H, A Short Remark on Fractional Variational Iteration Method, Physics Letters A, accepted, DOI: 10.1016 / j.physleta.2011.07.033

[5] He .J.H, Some asymptotic methods for strongly nonlinear equations, Int. J. Mod. Phys. B, 20(2006) 1141-1199

[6] Zhang .S, Exp-function Method: Solitary, Periodic and Rational Wave Solutions of Nonlinear Evolution Equations, Nonlinear Science Letters A, 1(2010)143-146

[7] Misirli. E, Gurefe .Y,The Exp-function Method to Solve the Generalized Burgers-Fisher Equation, Nonlinear Science Letters A, 1(2010) 323-328

[8] Zhang.S, Zong .Q, Liu. D, Gao .Q, A Generalized Exp-Function Method for Fractional Riccati Differential Equations, Communications in Fractional Calculus, 1(2010) 48-51

[9] Yang .X, Local Fractional Integral Transforms, Progress in Nonlinear Science, 4(2011) 1-225

[10] Yang .X, Local Fractional Integral Transforms, Progress in Nonlinear Science, 4(2011) 1-225 .

[11] Ganji .D. D, Seyed. H, Hashemi K, Analysis of Nonlinear Equations in Fluids, Progress in Nonlinear Science, 2(2011) 1-293

[12] Ganji .D. D, Seyed. H, Hashemi K, Analytical and Numerical Methods in Engineering and Applied Sciences, Progress in Nonlinear Science , 3(2011) 1-579.

[13] Gorenflo .R, Rutman .R, On ultraslow and intermediate processes," in Transform Methods and Special Functions, P. Rusev, I. Dimovski, and V. Kiryakova, Eds., pp. 61–81, Science Culture Technology Publishing, Singapore, 1995.

[14] Mainardi .F, Fractional relaxation and fractional diffusion equations, mathematical aspects," in Proceedings of the 12th IMACS World Congress, W. F. Ames, Ed., vol. 1, pp. 329–332, Georgia Tech Atlanta, 1994.

[15] Mainardi .F, Fractional calculus: some basic problems in continuum and statistical mechanics," in Fractals and Fractional Calculus in Continuum Mechanics, A. Carpinteri and F. Mainardi, Eds., pp. 291–348, Springer, New York, NY, USA, 1997.

[16] Ji-Huan He , S.K. Elagan, Z.B. Li, Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus, Physics Letters A 376 (2012) 257–259.